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SOME PROPERTIES OF A SMALL OPEN ECONOMY VERSION OF THE SOLOW-SWAN GROWTH MODEL

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Abstract

In this paper we examine some properties of the Benge and Wells (B-W) (1998a,b) formulation of a small open economy version of the Solow-Swan growth model. We suggest a simple method for classifying economies as capital importers or exporters (net debtors or creditors) in the steady state and highlight the simple mathematical structure of the B-W formulation. Given the small open economy assumption the fundamental dynamic equation for wealth is a simple first order linear differential equation so closed form solutions are immediate. That greatly simplifies the analysis of the transitional dynamics. In addition we show that the transitional dynamics of the B-W formulation depends on the specification of saving as a function of current income. In that formulation the speed of adjustment coefficient, β, is a function of both the saving ratio and the world real rate of interest. A debt trap is possible for some parameter values. When saving is specified as a function of permanent income (= steady state income) the speed of adjustment coefficient reverts to a constant determined by the exogenous growth rate in effective labour. The threat of a debt trap disappears.

JEL Classification: F41, F43, O41.

Key words. Solow-Swan, Growth model, Small Open Economy.

Some properties of a small open economy version of the Solow-Swan growth model

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I Introduction

In this paper we examine the properties of a small open economy version of the Slow-Swan model in continuous time. The model was proposed by Benge and Wells (B-W) (1998, a, b) and has the same structure as the discrete version of a small open economy Solow-Swan model developed by Milbourne (1997). The model exhibits non-degenerate dynamics for wealth and income in the small open economy case with perfect capital mobility and no adjustment costs.

As a small open economy is a price taker on world markets it treats the world rate of interest as given and can import or export capital. This feature means capital and output per effective worker can adjust instantaneously to their profit maximising values. It is also possible that the economy may be a net importer or exporter of capital (or, in the special case, neither an exporter or importer) in the steady state. In the B-W formulation the saving ratio plays a key role in determining which of these cases occurs and, in addition, pins down the speed of adjustment parameter in the analysis of the transitional dynamics. Consequently in this paper we focus on the key role of the saving ratio by suggesting a simple method of distinguishing between capital importers and exporters (net debtors and net creditors) in the steady state. We then consider the implications of an alternative specification of saving behaviour for the transitional dynamics.

In that regard the possible adjustment paths are more varied than the closed Solow-Swan model and depend both on the initial capital stock per effective worker relative to the steady state value and the specification of saving behaviour. We illustrate the analysis of the transitional dynamics for the case where saving is a constant fraction of current income and steady state capital per effective worker exceeds the initial value in the special case where the economy is neither a net importer or exporter of capital in the steady state. This is the case closest to the closed economy Solow-Swan model. The analysis is easily applied to illustrate other cases.

A key determinant of the results for the B-W formulation is the assumption about saving. If permanent income replaces current income in the specification of saving behaviour then the dynamics of the B-W formulation collapses in the sense that the speed of adjustment coefficient is no longer a function of the saving ratio. Instead it reverts to a constant determined by the exogenous growth rate in effective labour.

We also emphasise that the model has some attractive mathematical properties. In particular, the small open economy assumption with perfect capital mobility means that the fundamental differential equation for wealth is linear. Closed form solutions then

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follow directly which means that the analysis of the transitional dynamics is straightforward. It is not necessary to restrict the form of the production function to Cobb-Douglas or to take a linear approximation in the neighbourhood of the steady state as is the case with the closed economy Solow model (Barro and Sala-i-Martin, 1995). Nor is it necessary to resort to numerical methods as proposed by B-W (1998a, b).

II A general statement of the model

Following Benge and Wells (1998, a, b) we assume a model economy which produces a single commodity identical to that produced abroad. In an open economy, as a consequence of the possible import or export of capital and the associated interest payments, it is necessary to distinguish between production (or output) and income. In other words we begin with the distinction between *GDP* and *GNP* and the fundamental accounting identities in an open economy.

Let X represent *GNP*, Y real *GDP*, Z net exports, and F net foreign assets (if $F < 0$ the economy is a net importer of capital), C consumption and I investment. In an open economy, real $GDP = Y = C + I + Z$ and $GNP = X = Y + rF$ and as $X = C + S$ it follows that $S = I + Z + rF$. The distinction is important because in an open economy the fundamental accounting identity is then:

$$S = I + Z + rF \qquad (1)$$

Dynamics are introduced into the model through the growth in the capital stock and foreign assets. Net investment is defined in the following fashion;

$$I = \frac{dK}{dt} - D(K) \qquad (2)$$

where the usual assumption is that $D(K) = \delta K$ and δ is a constant proportional rate of depreciation. By definition the evolution of foreign assets is determined by;

$$\frac{dF}{dt} = Z + rF \qquad (3)$$

It is assumed that the aggregate production function, G satisfies the Uzawa-Inada conditions (Gandolfo (1996, p. 177), and exhibits constant returns to scale. Then,

$$Y(t) = G(K(t), E(t)) \qquad (4)$$

where, $K(t)$ is aggregate capital, and $E(t)$ is the number of effective workers. Full employment is assumed and growth is introduced by assuming that the number of effective workers grows as a result of population growth and technical progress determined exogenously at the rates n and μ respectively. Hence,

$$E = E_0 e^{(n+\mu)t}. \qquad (5)$$

Note that the introduction of labour-augmenting technological progress is necessary for the existence of a steady state solution in this model (Gandolfo, 1996, p. 183 or Barro and Sala-i-Martin, 1995, p. 54). The intention is to derive an equilibrium growth path such that K, E, Y, X, C and S all grow at the same rate; in this case at the rate: $n+\mu+\delta$. Per effective worker growth rates are zero in the steady state.

Having assumed constant returns to scale (homogeneity of degree one) the intensive form of the production function is written in lower case letters, i.e., $y = Y/E$ and $k = K/E$ so *GDP* per effective worker is

$$\frac{Y}{E} = G(\frac{K}{E},1) = y = g(k). \qquad (6)$$

Next, imposing the *small* open economy assumption and perfect capital mobility means that the rate of interest is exogenous. The desired capital stock per effective worker is then determined by the profit maximising condition

$$g'(k) = \delta + r \qquad (7)$$

where δ is the constant proportional rate of depreciation and r is the exogenous real world rate of interest. Assuming no adjustment costs and perfect capital mobility means that the capital and output per effective worker adjust immediately to their profit maximising levels.[1] Because in an open economy income may differ from output the adjustment to steady state values of income and wealth depends on the behaviour of net foreign assets which is in turn a function of the domestic saving ratio.

Given the profit maximising condition it follows that the steady state capital per effective worker is

$$k^* = g'^{-1}(\delta + r) \qquad (8)$$

and $y^* = g(k^*) = g(g'^{-1}(\delta + r))$. Also,

$$K(t) = g'^{-1}(\delta + r)E(t). \qquad (9)$$

and

$$Y = G(g'^{-1}(\delta + r),1)E(t). \qquad (10)$$

[1] Benge and Wells (1998a, b) also present an extended model in which they introduce adjustment costs but we don't consider that issue in this paper. Also see Barro and Sali-i-Martin (1995, p. 126) for an analysis of adjustment costs in the Solow-Swan model.

That is, the steady state capital per effective worker and *GDP* per effective worker are determined by the world real rate of interest and depreciation. Other parameters play no role.

It follows from the definition of gross investment, $I = \frac{dK}{dt} + \delta K$ that investment per effective worker is given by[2]

$$i = \frac{dk}{dt} + (\mu + n + \delta)k\,. \tag{11}$$

Hence in the steady state

$$i^* = (n + \mu + \delta)k^* \tag{12}$$

where k^* is given by expression (8). Note that this result is in contrast to the Solow-Swan model where the amount of saving out of *GDP* determines investment and hence the capital stock per effective worker. But in the small open economy the assumption of perfect capital mobility means that k jumps to k^*.

As noted above, the evolution of foreign assets is given by expression (3), $\frac{dF}{dt} = Z + rF$, which in intensive form appears as $\frac{df}{dt} = z + rf - (n + \mu)f$ [3] Substituting for z from the intensive form of the accounting identity $\hat{s} = i + z + rf$ where $\hat{s} = S/E$, produces

$$\frac{df}{dt} = \hat{s} - i - (n + \mu)f \tag{13}$$

Next we need some assumption about $\hat{s}$. Following Solow, Benge and Wells (1998,a,b) assume that a fixed proportion, s, of gross *current* income X is saved[4] Thus we have $S = sX$ or $\hat{s} = sx$. Hence we derive, by substitution for $\hat{s} = s(y + rf) = s(g(k) + rf)$ and i^*,

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[2] Make use of the fact that

$$\frac{dk}{dt} = \frac{d(\frac{K}{E})}{dt} = \frac{dK}{dt}\frac{1}{E} - (n + \mu)\frac{K}{E} so \frac{I}{E} = \frac{dK}{dE}\frac{1}{E} - (n + \mu)\frac{K}{E} + (n + \mu)\frac{K}{E} + \delta\frac{K}{E}$$ Which on substitution for dk/dt reduces to expression (11).

[3] As before, $\frac{dF/E}{dt} = \frac{dF}{dt}\frac{1}{E} - (n + \mu)\frac{F}{E}$.

[4] Milbourne (1997) employs a concept of permanent income. Below we examine the case where saving is a constant fraction of permanent income where the steady state value of wealth is treated as permanent income. The steady state solutions remain unchanged but the saving ratio no longer plays a role in the transitional dynamics.

$$\frac{df}{dt} + (n + \mu - sr)f = sg(k^*) - (n + \mu + \delta)k^* \tag{14}$$

which is the fundamental differential equation for foreign assets. Because in the small open economy k^* is fixed, expression (14) is a simple linear first order differential equation so the equilibrium solution and stability condition follow directly[5]

The particular solution provides the steady state value of f as

$$f^* = \frac{sg(k^*) - (n + \mu + \delta)k^*}{n + \mu - sr} \tag{15}$$

The stability requirement $(n + \mu - sr) > 0$ follows immediately from examination of the complete solution which for $n + \mu - sr \neq 0$ is

$$f(t) = [f^* - f(0)]e^{-(n+\mu-sr)t} + f^* \tag{16}$$

and for $n + \mu - sr = 0$ is

$$f(t) = f(0) + (sg(k^*) - (n + \mu + \delta)k^*)t\,. \tag{16'}$$

As Milbourne (1997) suggests, the stability condition has an economic interpretation[6] In the context of this version of the model it is apparent that the higher the exogenous real rate of interest, r, the greater the likely hood that no steady state solution exists as the import or export of capital is unbounded. The assumptions underlying the small open economy simply breakdown. Given the values of the other parameters the critical range for the interest rate occurs for any $r \geq \frac{n + \mu}{s}$. It is also apparent that in this specification of the model the saving ratio has the same effect. Given the other parameters in the model any saving ratio $s \geq \frac{n + \mu}{r}$ is not compatible with the existence of a steady state as capital import or export is unbounded[7] In what follows we concentrate on the steady state solutions.

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[5] A linear first-order differential equation of the form $\frac{df}{dt} + af = b$ where a and b are constants, has a particular solution $f_p = b/a$ and a general solution of the form $f(t) = Ae^{-at} + b/a;\ a \neq 0$ and if $a = 0$ then $f(t) = A + bt$. The arbitrary constant can be determined by the initial conditions.

[6] For a reconciliation between Milbourne's (1997) version of the stability condition and the version in this paper see B-W (1998, appendix).

[7] The unbounded capital import outcome corresponds to the case referred to in the literature as the debt-trap scenario. That scenario is usually associated with a saving ratio that is too low as a high saving ratio is thought to obviate the need for foreign capital. But in this model with perfect capital mobility a high saving

As a convenient means of distinguishing the capital importer (net debtor) from the capital exporter (net creditor) in the steady state find the value of s for which $f^* = 0$. This value of s follows directly from (15) as

$$\bar{s} = (n + \mu + \delta)\frac{k^*}{g(k^*)} \tag{17}$$

Using this result we can re-write expression (15) as

$$f^* = \frac{(s - \bar{s})g(k^*)}{n + \mu - sr}. \tag{17'}$$

The steady state value of income, x^* is derived directly from the definition $x^* = y^* + rf^*$ and expressions (15) and (17) to produce

$$x^* = \left(\frac{n + \mu - \bar{s}r}{n + \mu - sr}\right)g(k^*). \tag{18}$$

Expression (18) clearly indicates the relationship between *GNP* and *GDP* for different values of s. Recall that if $s = \bar{s}$ the economy has a zero net foreign asset position in the steady state and $sx^* = sy^*$ From (17') it is also immediately apparent that $f^* < 0$ if $s < \bar{s}$ (in a stable system). That is, the economy is a capital exporter (net creditor) in the steady state if $s > \bar{s}$ and a capital importer (net debtor) in the steady state is $s < \bar{s}$. This innovation provides a useful method for classifying economies which we apply in section III.

The steady state value of consumption follows directly from the consumption function $c^* = (1 - s)x^*$ which, by applying (17), may be written as

$$c^* = (1 - s)\left(\frac{n + \mu - \bar{s}r}{n + \mu - sr}\right)g(k^*). \tag{19}$$

Define wealth per effective worker as

$$w = k + f \tag{20}$$

As the steady state capital stock is tied down by the exogenous real rate of interest the dynamic behaviour of w and f is simply related. This means that we can write[8]

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ratio means instantaneous adjustment to the profit maximising value of capital per effective worker even if it turns out that no steady state values for income, wealth and foreign debt (= capital import) exist. The latter outcome suggests myopic behaviour.

[8] Expression (21) follows in straightforward fashion from

$$\frac{dw}{dt} + (n + \mu - sr)w = s(g(k^*) - rk^*) - \delta k^* \tag{21}$$

which by the previous analysis yields the steady state solution for w^* as

$$w^* = \frac{s(g(k^*) - rk^*) - \delta k^*}{n + \mu - sr} \tag{22}$$

and as expected the stability condition is $(n + \mu - sr) > 0$.

Finally the steady-state value for net exports per effective worker, z, is derived in similar fashion from the intensive form of the evolution of foreign assets $\frac{df}{dt} + (n + \mu - r)f = z$, and in a steady state reduces to

$$z^* = (n + \mu - r)f^* \tag{23}$$

That completes the steady state solution of the model.

III Graphical illustration of the steady states of the model

The form of presentation of the model in the previous section is motivated by the need to remain as close as possible to the Solow formulation while at the same time providing for a simple graphical interpretation. With these objectives in mind it is useful to follow B-W (1998a) and begin with the case of zero net foreign assets in the steady state as this is the case closest to the Solow model of a closed economy.

Case 1 *Steady state when $s = \bar{s}$ so $f^* = 0$* - (neither a lender nor a borrower be).
This is the case in which the domestic saving ratio is such that the economy does not engage in the import or export of capital (foreign lending or borrowing) in the steady state so that wealth per effective worker consists of the stock of domestic capital per worker only. From (14) and (17') when $s = \bar{s}$ the dynamic equation for f is the homogenous case

$$\frac{df}{dt} + (n + \mu - sr)f = 0. \tag{24}$$

$$\frac{dw}{dt} + (n + \mu - sr)(w - k^*) = sg(k^*) - (n + \mu + \delta)k^*.$$

In this case the steady state value for f is obviously zero. Inspection of (15) shows that f^* $= 0$ implies that $sg(k^*) - (n+\mu+\delta)k^* = 0$ for a stable system. In other words, given the steady state value of k the domestic saving rate is just sufficient to maintain that value given the population growth rate, productivity growth and rate of depreciation.

Graphically, this case is illustrated in Figure 1 where, following B-W (1998a), wealth and capital per effective worker are measured on the horizontal axis with the distance between them indicating the value of f in terms of the definition of wealth. The vertical axis measures real *GDP* and *GNP* per effective worker.

The steady state capital stock per effective worker, k^* and *GDP* per effective worker, y^* are determined by the world real rate of interest (allowing for depreciation) in terms of the profit maximising condition (7). This condition remains unchanged for all cases given the world real rate of interest and is not influenced by any other parameters in the model.

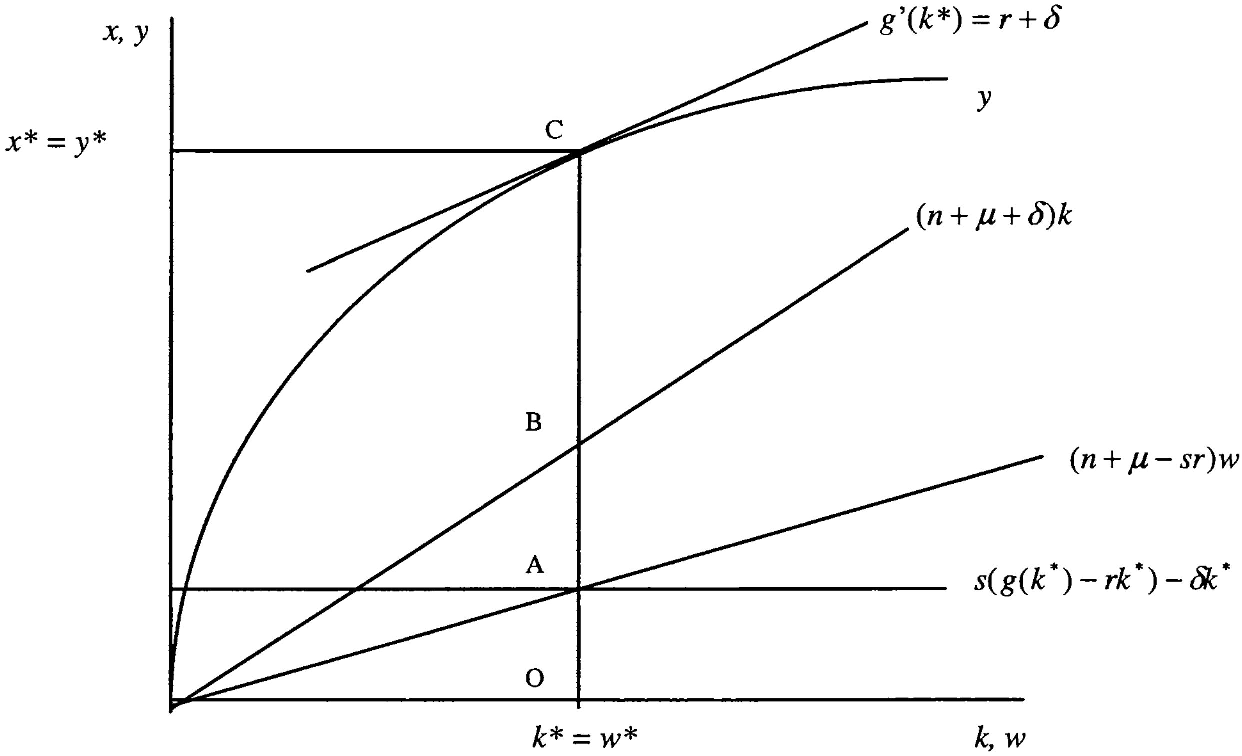

Figure 1 $s = \bar{s}$ *so* $f^* = 0$ *and* $w^* = k^*$.

The steady state value of wealth per effective worker, w^*, is determined by expression (22) for the given values of k^* and y^*. The graphical version of expression (22) is illustrated by drawing the numerator on the RHS as the horizontal line labelled $s[g(k^*) - rk^*] - \delta k^*$ and a ray from the origin labelled $(n+\mu-sr)w$. The intersection of these lines gives w^* which for this case occurs at $w^* = k^*$. If $s[g(k^*) - rk^*] - \delta k^* >$

$(n+\mu-sr)w$ then net saving is greater than is required to keep wealth per effective worker constant and $dw/dt > 0$.

With reference to Figure 1 the distance OB = i^*, while the distance BC = c^*. As $f^* = 0$, $z^* = 0$ also (from (23)) so we have the simple national accounting identity, $y^* = c^* + i^*$. Note as $x^* = y^* + rf$, $y^* = x^*$ in this case.

Figure 1 is directly comparable to the Solow-Swan model of a closed economy but the small open economy version differs in one crucial respect. In the small open economy, the steady state solution for y^* and k^* is determined by the exogenous real rate of interest and not by the saving ratio, s, as would be the case in a closed economy. Capital flows immediately to maintain the profit maximising condition (7) and capital and output per effective worker adjust immediately to their steady state values. This feature of the model may well be relevant to the analysis of convergence across states or regions with a national identity but may be less reasonable for the analysis of convergence across countries.

Case 2 *Steady state when* $s > \bar{s}$ and $f^* > 0$ (Capital exporter in the steady state).
In the small open economy version of the model the saving ratio determines whether the economy is a capital importer (net debtor) or a capital exporter (net creditor) in the steady state. For example, if $s > \bar{s}$ the economy will accumulate foreign capital in addition to domestic capital and the steady state value of wealth per effective worker will exceed the value of the domestic capital stock per effective worker. In such a steady state saving must be sufficient to keep capital and wealth constant in effective worker terms.

In the steady state the economy earns income on its holding of foreign capital so steady state *GNP*, x^* , is greater than steady state *GDP*, y^*. Graphically, this means that x^* lies vertically above y^* by the distance rf^* (refer to the definition of x). When $s > \bar{s}$ the steady state value of wealth per effective worker, w^*, as determined by expression (22), is greater than the steady state value of capital per effective worker, k^* as $f^* > 0$. In this case, the graphical solution of expression (22) as illustrated in Figure 2, gives the point of intersection A of the horizontal line $s(g(k^*) - rk^*) - \delta k^*$ and the ray $(n+\mu-sr)w$ at $w^* > k^*$ as $f^* > 0$.

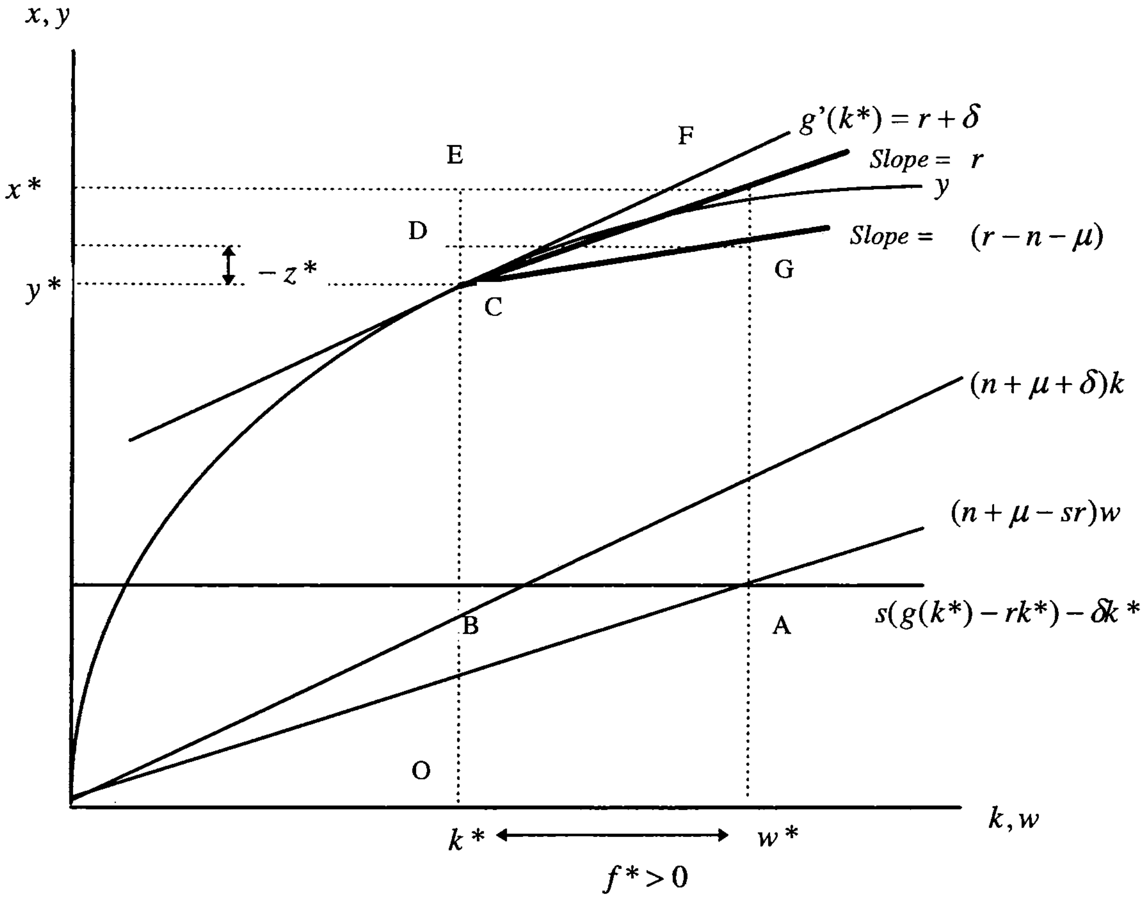

Figure 2 Steady state for $s > \bar{s}$, $f^* > 0$ and $n + \mu < r$

The distance between k^* and w^* is f^* by definition. As noted above, Benge and Wells (1998a) point out that in the steady state saving must be large enough to maintain both domestic and foreign capital constant in effective-worker terms. This can be seen by re-arrangement of expression (15) to obtain

$$s(g(k^*) + rf^*) = (n + \mu + \delta)k^* + (n + \mu)f^* \tag{25}$$

Expression (25) is the open economy equivalent of the fundamental equation of the Solow-Swan model which collapses to *Case 1* above when $s = \bar{s}$ as $f^* = 0$. The national accounting identity then follows by substituting expression (25) in the identity $y^* = (1 - s)(y^* + rf^*) + s(y^* + rf^*) - rf^*$ to produce $y^* = (1 - s)(y^* + rf^*) + (n + \mu + \delta)k^* + (n + \mu - r)f^*$ which is simply the accounting identity, $y^* = c^* + i^* - z^*$.

The distance z^* is obtained in the diagram by drawing in a line with slope $r - n - \mu$ originating at point C and intersecting the vertical line above w^* at point G (assuming $n + \mu < r$). By simple geometry the distance CD is net imports (-z*). In other words this economy is running a trade deficit financed by a net income surplus from foreign earnings on its stock of foreign capital.

The other solid line originating at point C and drawn with slope r simply illustrates that the distance CE represents rf^*. The component CD is consumed while the remainder is saved to maintain domestic and foreign capital constant in effective worker terms. The distance OB represents steady state investment, i^*, while BD is consumption, c^* and CD is net imports (- z^*) so the national accounting identity reads OC = BD + OB + CD; y* = c* + i* - z*.

Case 3 Steady State when $s < \bar{s}$ *and* $f^* < 0$ (Capital importer in the steady state).
For the sake of completeness Case 3 $s < \bar{s}$ is illustrated in Figure 3. The economy is a capital importer (net debtor) in the steady state, and income (*GNP*) is less than *GDP* . In this case the national accounting identity reads OC = BD + OB + DC; $y^* = c^* + i^* + z^*$.

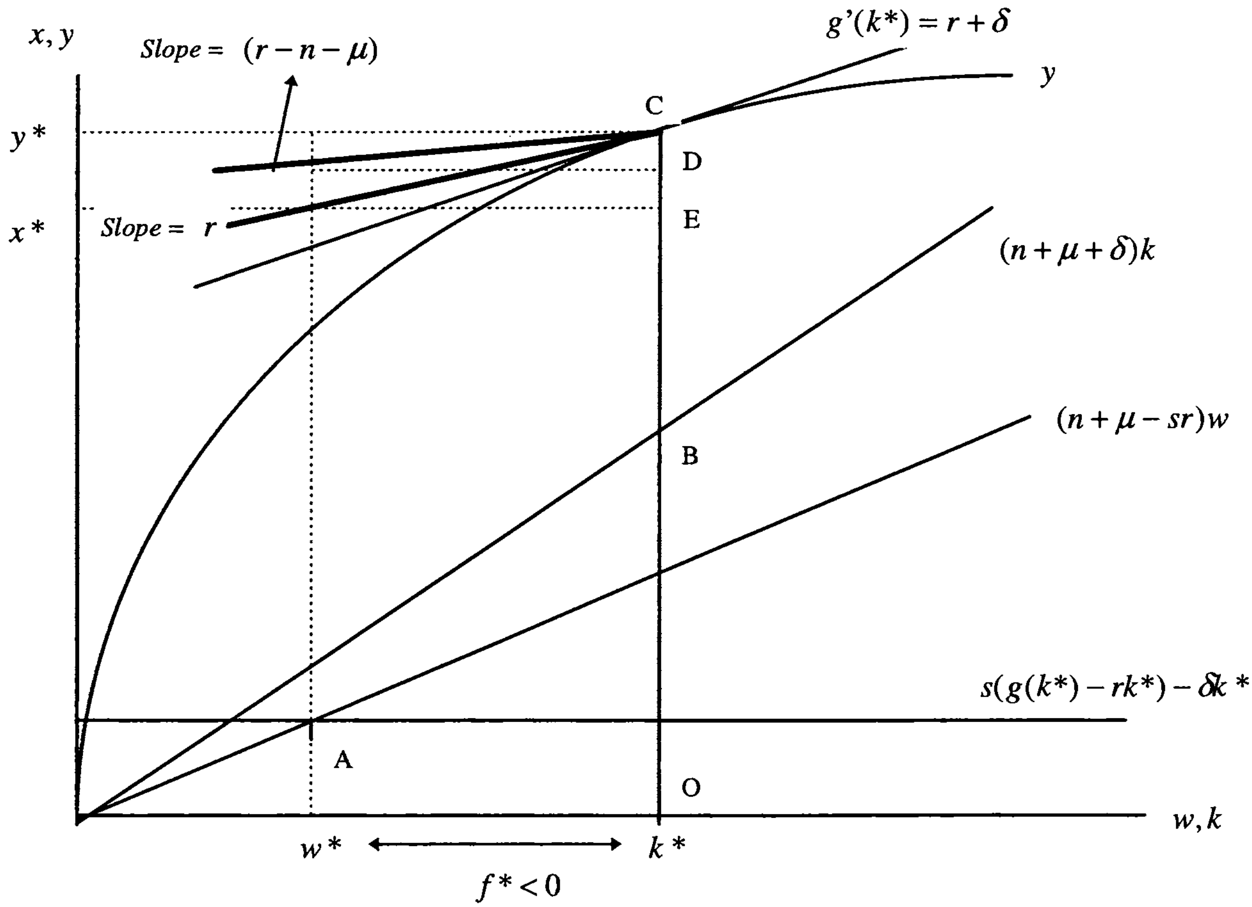

Figure 3 Case 3 Steady state for $s < \bar{s}$, $f^* < 0$ and $n + \mu < r$.

IV Transitional Dynamics

In many respects the transitional dynamics for wealth per effective worker in the B-W model mimics that for capital per effective worker in the Solow-Swan model on which it is based. For example, the fundamental differential equation for wealth is given by (21) so we can plot the horizontal line $(n+\mu-sr)$ and the curve $\dfrac{s(g(k^*)-rk^*)-\delta k^*}{w}$ in similar fashion to that suggested by by Barro and Sala-i- Martin (1995) for capital per effective worker in the Solow-Swan model. Figure 4 illustrates the transitional dynamics for wealth in the B-W model.

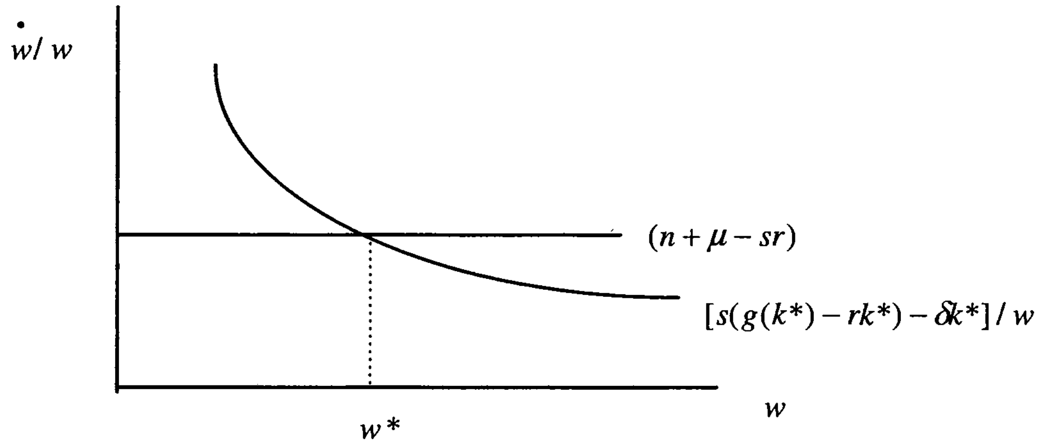

Figure 4 Transitional dynamics for wealth in the B-W model

Inspection of expression (14) reveals that the transitional dynamics for foreign assets is similar to that for wealth. The traditional distinction between conditional and absolute converge therefore carries over to the B-W model and in that case it is instructive to examine the transitional dynamics of the B-W model in more detail.

As there are three types of steady state solution in the B-W formulation of the model (as illustrated in section III) to simplify matters we consider the case of $s=\bar{s}$ and $f^*=0$ in the steady state. In this case the solutions for net foreign assets and wealth per effective worker are given by expression (26) and (27) respectively (given the linear form of the B-W fundamental equation);

$$f(t) = f(0)e^{-(n+\mu-sr)t} \tag{26}$$

$$w(t) = k^* + f(0)e^{-(n+\mu-sr)t} \tag{27}$$

To highlight the nature of the transitional dynamics consider the case where $f(0)<0$ and $f^*=0$. During the transition to the steady state this economy is importing capital , i.e., it is a net debtor, but as time elapses the rate of saving out of current income per effective worker is greater than is required to maintain the initial level of wealth per effective worker and wealth per effective worker converges on its steady state value, equivalent in this case, to the steady state value of capital per effective worker as illustrated in Figure 1. Graphically, the initial situation is illustrated in Figure 5 below. Initially, and during transition, income, x, is depressed below output, y, by the interest payment, rf, on the imported capital.

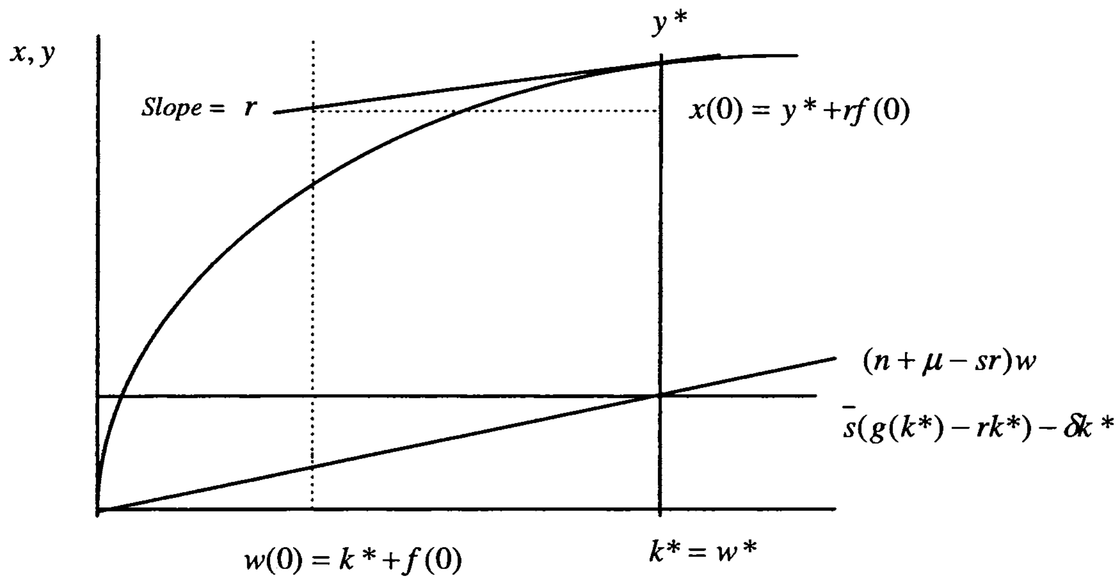

Figure 5 Initial capital import for the case $f(0)<0$, $w(0) > 0$ and $s = \bar{s}$ so $k^* = w^*$.

Figure 6 below then illustrates the dynamic behaviour of f and w over time for a particular case of the initial condition $f(0) < 0$ and the case where $s = \bar{s}$ so $f^* = 0$ and $w^* = k^*$. Similar exercises can be undertaken for steady states in which the economy is a capital importer or exporter and for the alternative initial condition $f(0) > 0$.

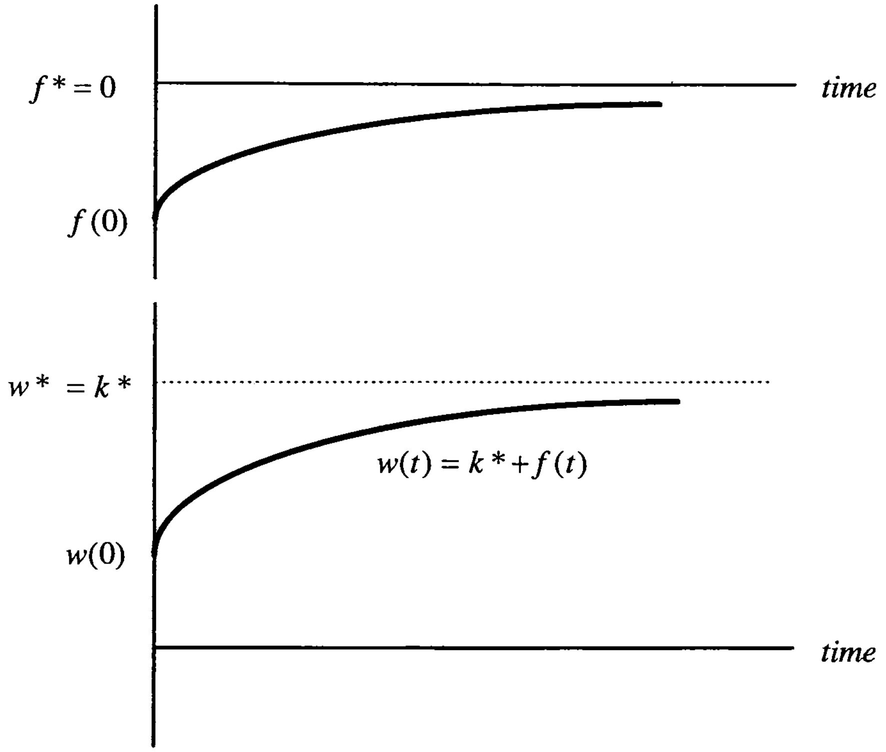

Figure 6 The dynamic adjustment paths for f and w in the open economy Solow-Swan model for the case $f(0) < 0$, $w(0) > 0$ and $w^* = k^*$.

To examine the transitional dynamics of the model, B-W (1998, a) use numerical methods in the context of a discrete version of the model to determine the speed of convergence of income, x and wealth, w, per effective worker to the steady state values.

However in view of the linear form of the fundamental equation for the B-W formulation of the model we suggest that the use of numerical methods is unnecessary as the half-life can be calculated directly from the solution to the model. The transitional dynamics can be analysed formally without resorting to linear approximation in the vicinity of the steady state or reliance of the Cobb-Douglas form of technology. The differential equation describing the dynamics of the model is linear so the solution is immediate. For the B-W formulation the general solution to the differential equation describing the behaviour of wealth, expression (21), is simply given by

$$w(t) = \frac{s[g(k^*) - rk^*] - \delta k^*}{n + \mu - sr} + \left[w(0) - \frac{s[g(k^*) - rk^*] - \delta k^*}{n + \mu - sr} \right] e^{-(n+\mu-sr)t} \tag{28}$$

which is equivalent to,

$$w(t_\theta) = w^* + [w(0) - w^*] e^{-(n+\mu-sr)t_\theta} \tag{29}$$

from which, following Gandolfo (1996), the half-life is[9],

$$t_{0.5} = -\ln 0.5 / (n + \mu - sr).$$

Thus in the case of the B-W formulation of the model the speed of convergence parameter, $\beta = (n + \mu - sr)$ is unambiguously constant irrespective of the distance from the steady state. Also, unlike the Solow -Swan model the speed of convergence parameter in the B-W formulation of the model is a function of s. Figure 6 illustrates the half-life as a function of s in the B-W formulation.

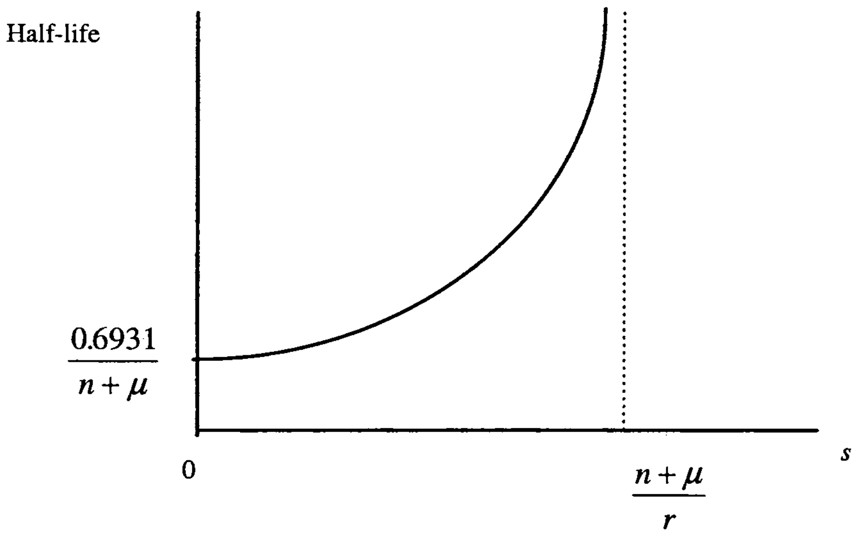

Figure 7 Half-life in B-W model as a function of saving ratio, s.

As is apparent from Figure 7, in this case, because the speed of adjustment coefficient, $\beta = (n + \mu - sr)$ is inversely related to s, the half-life is an increasing function of s with an asymptote at $\frac{n+\mu}{r}$. By comparison, the speed of adjustment coefficient in the closed economy Solow-Swan model is given by the parameter $\beta = (1-\alpha)(n + \mu + \delta)$ which is a constant and is independent of the saving ratio s. Notice that in the B-W formulation of the open economy version of the model the half-life tends to become infinitely large as the saving ratio approaches the value at which the importing and exporting of capital is no longer bounded and per effective worker steady states do not exist.

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[9] Following Gandolfo (1995, p.185) we can write: $w(t_\vartheta) = w(0) + \vartheta(w^* - w(0))$ where t_ϑ is the time it takes for the model to complete some fraction ϑ of the gap between $w(0)$ and w^* and $0 < \vartheta < 1$. For the half-life $\vartheta = 1/2$. Applying this information to the general solution $w(t_\vartheta) = w^* + [w(0) - w^*]e^{-\beta t_\vartheta}$ which on rearranging to solve for t_ϑ we get $e^{-\beta t_\vartheta} = 1 - \vartheta$ so that

$$t_\vartheta = \frac{-\ln(1-\vartheta)}{\beta}$$

Finally, we briefly review some of the empirics associated with the analysis of the transitional dynamics. To get some idea of the adjustment time to the steady state consider the parameter values suggested by B-W (1998a); viz, $\mu = 0.02; n = 0.01; r = 0.1; \delta = 0.04$. Together with a capital output ratio (*K/Y*) of 2.4 these parameters imply a value for $\bar{s}$ of 0.168 from expression (16). This is the case for which $f^* = 0$, i.e., a world of zero foreign ownership in the steady state. Applying the three saving ratios corresponding to the three cases illustrated in section III produces the following results[10]

Case 1: $s = \bar{s} = 0.168$ and $f^* = 0$

$$\bar{t}_{0.5} = -\ln 0.5 / (0.03 - 0.0168) = 0.69315 / 0.0132 = 52.51 \text{ years.}$$

Case 2: $s = 0.2 > \bar{s}$ and $f^* > 0$.

$$\bar{t}_{0.5} = -\ln 0.5 / (0.03 - 0.02) = 0.69315 / 0.01 = 69.31 \text{ years.}$$

Case 3: $s = 0.1 < \bar{s}$ and $f^* < 0$.

$$\bar{t}_{0.5} = -\ln 0.5 / (0.03 - 0.01) = 0.69315/0.02 = 34.66 \text{ years.}$$

We note in passing that these results for Cases 1 and 2 are significantly higher than those reported by B-W (1998, b) in their Figure 4. The B-W (1998,b) results from their discrete version of the model using numerical methods are as follows: for Case 1 (s = *0.168*), $\bar{t}_{0.5} = 49.8$ and the result for Case 2 ($s = 0.2$) $\bar{t}_{0.5} = 60$. The result for Case 3 ($s = 0.1$) is $\bar{t}_{0.5} = 35$ which corresponds to our result. This suggests an increasing approximation error in the B-W discrete version of the model for values of $s > 0.1$.

V Specification of saving behaviour

Before leaving the properties of the transitional dynamics we briefly examine what impact the change to the specification of saving behaviour has on the properties of the B-W version of the model. This is an issue because Milbourne (1997) assumes that consumers save a constant proportion of *permanent* income while B-W (1998) assume that consumers save a constant proportion of *current* income .

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8. A half-life of 70 means that it would take over 100 years (105 years) to get 75% of the way to the steady state. Barro and Sala-i-Martin (1995, p. 38) suggest that a β coefficient (the denominator in the above calculations) between 1.5 and 3.0 per cent per year is consistent with the data (the 2 per cent per year "rule"). Note that the β coefficient calculated by Barro and Sala-i-Martin (1995) refers to convergence to the steady state per effective worker value $k^{1-\alpha}*$.

Presumably, permanent income in the B-W model is the steady state value $x^* = y^* + rf^*$ so the saving function is simply; $\hat{s} = sx^* = s(y^* + rf^*)$. The differential equation for net foreign assets is then;

$$\frac{df}{dt} + (n+\mu)f = s(g(k^*) - rf^*) - (n+\mu+\delta)k^* \tag{30}$$

which for the usual case of $(n+\mu) > 0$ produces the general solution;

$$f(t) = Ae^{-(n+\mu)t} + \frac{s(g(k^*) + rf^*) - (n+\mu+\delta)k^*}{(n+\mu)} \tag{31}$$

The steady state solution remains unchanged but it is apparent that the speed of adjustment parameter, β, is no longer a function of s. Hence the threat of unbounded import or export of capital does not arise in the model if saving is a function of permanent rather than current income. In this case $\beta = (n+\mu)$ and for the parameter values $\mu = 0.01$ and $n = 0.02$ the half-life is given by $\overline{t_{0.5}} = 0.69315/0.03 = 23$ years which is somewhat higher that the result for an equivalent closed Solow-Swan model of 14.8 years presented by B-W (1998, p. 18).

VI Concluding remarks

The key feature of the B-W formulation of the open economy Solow-Swan model is that the transitional dynamics are not degenerate even if capital and output per effective worker jump immediately to their steady state values (assuming existence). Income and wealth adjust slowly and parameter values suggested by B-W indicate that convergence to the steady state in the small open economy Solow-Swan model is slower than closed version of the model.

In this regard an important property of the B-W formulation is the role of the saving rate in defining the nature of the transitional dynamics. The B-W assumption that saving is a constant fraction of *current* income is closest to the Solow-Swan specification and allows the saving ratio to play this crucial role. For this formulation we suggest a simple method for classifying economies as capital importers (net debtors) or exporters (net creditors) in the steady state. This formulation of the model also predicts that open economies with high saving ratios adjust far more slowly to the steady state than do economies with lower saving ratios. This feature of the model is due to the asymptotic analysis of stability and may be an inherent limitation of the B-W formulation of the model. However, the key role of the saving ratio in the transitional dynamics disappears if saving is a constant fraction of *permanent* rather than *current* income. When the steady state value of income is taken as a proxy for permanent income, which seems reasonable in the context of the model, the dynamics is far less interesting although the range of possible steady states remains unchanged. The risk of debt-traps and explosive capital accumulation disappears.

On the technical side the mathematics of the small open economy version of the model is simpler than the traditional Solow model because the exogenous real interest rate and the assumption of a perfect capital market produces a linear form for the fundamental equations for wealth or net foreign assets. Closed form solutions are then immediate. Therefore it is not necessary to resort to linear approximation, use of the Cobb-Douglas production function or numerical approximation to examine the properties of the model.

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